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BOUNDING SOLUTIONS FOR CREEPING STRUCTURES SUBJECTED TO LOAD VARIATIONS ABOVE THE SHAKEDOWN LIMIT

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Abstract—For load variations above the shakedown limit, cyclic plasticity solutions are defined for yield criteria of perfect-plasticity and of kinematic strain-hardening. The cyclic plasticity solutions are used to provide upper bounds on the work, displacements and creep energy dissipations which occur in the cyclic stationary state of a creeping structure.

NOTATION

δ_{ij}	Kronecker's delta
ϵ_{ij}	total strain tensor
θ, θ_{ij}	temperature and thermal strain tensor, respectively
$\dot{\lambda}$	plastic multiplier
ρ	positive-definite metric
σ_{ij}, σ_0	stress tensor and constant stress, respectively
τ	cycle time
$\phi(\sigma_{ij}/\sigma_0)$	homogeneous function of degree one in σ_{ij}/σ_0
A_{ijkl}	tensor of elastic constants
$\dot{D}$	creep energy dissipation rate
e_{ij}	elastic strain tensor
$f(\sigma_{ij})$	yield function
$g(\theta)$	positive function of temperature
k, m, n	constants
P_i	applied loads
p_{ij}	plastic strain tensor
R_i	additional loads
S	surface
S_{ij}	stress deviator tensor $\{=\sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}\}$
T_{ij}	tensor $T_{ij} = S_{ij} - mp_{ij}$
t	time
u, u_i	displacement
V	volume
V_{ij}, V_0	creep strain tensor and constant strain, respectively
x_i	rectangular co-ordinates
$(\dot{\ })$	$\partial(\ )/\partial t$.

1. INTRODUCTION

The paper considers structures which operate in the creep range and which are subjected to periodic variations of temperature and external loading. For load and temperature variations below shakedown, Ponter[1, 2] has shown that a stress history which is the sum of the periodic elastic solution and a constant residual stress field can be used to provide upper bounds on creep energy dissipation and displacements for a structure in the cyclic stationary state. The optimum stress history used to bound the creep energy dissipation is of importance since it approaches the actual stress history when cycle times become short compared to material time scales[2, 3]. In many practical situations cycle times are short compared to characteristic relaxation times of structures and the optimal stress history may be considered as an approximate solution. It may be used to relate average deformation rates of structures to constant load, constant temperature uniaxial data by means of reference stresses and reference temperatures[4].

For loading above the shakedown limit, it is no longer possible to find a solution which is the sum of the elastic solution and a constant residual stress and which does not violate yield at some time. Consequently, the existing methods outlined above cannot be applied in this area. In situations where the loading is purely mechanical, the limitations of shakedown can usually be satisfied by suitable design and present methods are adequate. However, in some nuclear power applications, severe temperature gradients occur and it is difficult to avoid violating shakedown.

If an expensive, full inelastic analysis is to be avoided, it is then necessary to extend previous work. In the present paper, the bounding method of Ponter[1] is extended to cater for load and temperature variations above the shakedown limit. Attention is restricted to periodic loading and to the behaviour of the structure in the cyclic stationary state. Any transient response in reaching this cyclic stationary state is not considered. The use of the optimum stress history for creep energy dissipation as an approximate solution is discussed. Application of the bounds to some simple structures is considered in a companion paper[5].

2. ASSUMPTIONS AND MATERIAL BEHAVIOUR

Consider a body of volume, V, surface, S, with negligible body forces. This is subject to a given history of loading $P_i(t)$ over part S_p of S and to zero surface velocities over the remainder S_u. The temperature, $\theta(x_i, t)$, is a given function of position, x_i, and time, t. All deformations are assumed small so that changes in geometry may be neglected.

The total strain rate $\dot{\epsilon}_{ij}$ is considered as the sum of four parts

$$\dot{\epsilon}_{ij} = \dot{e}_{ij} + \dot{V}_{ij} + \dot{p}_{ij} + \dot{\theta}_{ij} \tag{1}$$

where $\dot{e}_{ij}$, $\dot{V}_{ij}$, $\dot{p}_{ij}$, $\dot{\theta}_{ij}$ are elastic, creep, plastic and imposed (or thermal) strain rates respectively. In practice, it is difficult to separate creep and plastic strains and the distinction in eqn (1) is introduced for convenience. Plastic deformation is assumed to be instantaneous (time-independent) whereas creep deformation is assumed to occur over a period of time. Interactions such as the effect of creep on the yield stress and the effect of plastic strains on the creep strain rates are not considered.

Elastic strains are related to the stress tensor σ_{kl} by a generalised Hooke's Law,

$$e_{ij} = A_{ijkl}\sigma_{kl} \tag{2}$$

where the tensor of elastic constant A_{ijkl} has the usual symmetry properties. The creep energy dissipation rate per unit volume, $\dot{D}$, is taken as

$$\dot{D}(\sigma_{ij}/\sigma_0) = \sigma_{ij}\dot{V}_{ij} = \sigma_0\dot{V}_0\phi^{n+1}(\sigma_{ij}/\sigma_0)g(\theta) \tag{3}$$

where n, σ_0, $\dot{V}_0$ are constants and $g(\theta)$ is a positive function of temperature, θ. The function ϕ is convex and homogeneous of degree one in (σ_{ij}/σ_0) and reduces to the value unity for a uniaxial stress σ_0. Following Calladine and Drucker[6] creep strain rates are normal to surfaces of constant energy dissipation rate and are given by

$$\dot{V}_{ij}/\dot{V}_0 = \phi^n\{\partial\phi/\partial(\sigma_{ij}/\sigma_0)\}g(\theta). \tag{4}$$

Since ϕ is convex and homogeneous of degree one in (σ_{ij}/σ_0), it readily follows that any two states of stress σ_{ij}, σ_{ij}^* and the corresponding creep strain rates $\dot{V}_{ij}$, $\dot{V}_{ij}^*$, at the same temperature θ, must satisfy (see, for example, Martin[7])

$$\begin{aligned} &(\sigma_{ij}^* - \sigma_{ij})(\dot{V}_{ij}^* - \dot{V}_{ij}) \geq 0 \\ &(n+1)(\sigma_{ij}^* - \sigma_{ij})\dot{V}_{ij} \leq \dot{D}(\sigma_{ij}^*/\sigma_0) - \dot{D}(\sigma_{ij}/\sigma_0) \\ &\quad n(\sigma_{ij}^* - \sigma_{ij})\dot{V}_{ij} \leq (n/n+1)^{n+1}\dot{D}(\sigma_{ij}^*/\sigma_0). \end{aligned} \tag{5}$$

Two types of plastic behaviour are considered. The first is one of perfect plasticity with yield criterion $f(\sigma_{ij}) \leq 0$ where the yield surface $f(\sigma_{ij}) = 0$ is strictly convex. The associated flow rule is

$$\dot{p}_{ij} = \dot{\lambda}\partial f/\partial\sigma_{ij} \tag{6}$$

where $\dot{\lambda} = 0$ if $f < 0$ or if $f = 0$ and $\{\partial f/\partial\sigma_{ij}\}\dot{\sigma}_{ij} < 0$

$$\dot{\lambda} \geq 0 \quad \text{if } f = 0 \qquad \text{and} \qquad \{\partial f/\partial\sigma_{ij}\}\dot{\sigma}_{ij} = 0.$$

The convexity of the yield surface and the use of the associated flow rule have the following consequences (Koiter[8]): (a) if σ_{ij}, $\dot{p}_{ij}$ and σ_{ij}^*, $\dot{p}_{ij}^*$ are two pairs of allowable states of stress and associated plastic strain rates

$$(\sigma_{ij}^* - \sigma_{ij})(\dot{p}_{ij}^* - \dot{p}_{ij}) \geq 0 \tag{7}$$

where equality holds if, and only if, $\sigma_{ij} = \sigma_{ij}^*$ or $\dot{p}_{ij} = \dot{p}_{ij}^* = 0$, (b) if $\dot{\sigma}_{ij}$, $\dot{p}_{ij}$ and $\dot{\sigma}_{ij}^*$, $\dot{p}_{ij}^*$ are two pairs of stress rates and associated plastic strain rates in the same state of stress on the yield surface

$$(\dot{\sigma}_{ij}^* - \dot{\sigma}_{ij})(\dot{p}_{ij}^* - \dot{p}_{ij}) \geq 0. \tag{8}$$

The second type of plastic behaviour considered is one which has linear kinematic strain hardening. The model used, a simplication of one proposed by Edelman and Drucker[9], has yield criterion

$$T_{ij}T_{ij} - k^2 \leq 0$$

where

$$T_{ij} = S_{ij} - mp_{ij}. \tag{9}$$

Here, m and k are positive material constants. The effect of temperature on m and k is not considered. The associated flow rule is

$$\dot{p}_{ij} = \dot{\Lambda} T_{ij} \tag{10}$$

where

$$\dot{\Lambda} = 0 \quad \text{if } T_{ij}T_{ij} < k^2 \quad \text{or} \quad \text{if } T_{ij}T_{ij} = k^2 \quad \text{and} \quad T_{ij}\dot{S}_{ij} < 0$$

$$\dot{\Lambda} = T_{ij}\dot{S}_{ij}/mk^2 \quad \text{if } T_{ij}T_{ij} = k^2 \quad \text{and} \quad T_{ij}\dot{S}_{ij} \geq 0.$$

Consequences of the yield criterion and associated flow rule are (a) if T_{ij}, $\dot{p}_{ij}$ and T_{ij}^*, $\dot{p}_{ij}^*$ represent two states of stress, plastic strain and associated plastic strain rates,

$$(T_{ij}^* - T_{ij})(\dot{p}_{ij}^* - \dot{p}_{ij}) \geq 0 \tag{11}$$

where equality holds if, and only if, $T_{ij} = T_{ij}^*$ or $\dot{p}_{ij} = \dot{p}_{ij}^* = 0$, (b) if $\dot{T}_{ij}$, $\dot{p}_{ij}$ and $\dot{T}_{ij}^*$, $\dot{p}_{ij}^*$ represent two pairs of stress rates and associated plastic strain rates for which $T_{ij} = T_{ij}^*$,

$$(\dot{T}_{ij}^* - \dot{T}_{ij})(\dot{p}_{ij}^* - \dot{p}_{ij}) \geq 0. \tag{12}$$

3. CYCLIC STATIONARY STATES

Frederick and Armstrong[10] have shown that structures composed of perfectly-plastic creeping material settle down to a periodic stress distribution when subjected to periodic load variations. A number of authors (Ponter[1]; Mroz[11]; Boyle[12]) have extended this result to a wider class of material behaviour. In this section, the result is extended slightly to include kinematic hardening plastic materials. Particular attention is paid to the properties of the cyclic stationary state when there is no creep.

Consider a body of volume V, surface S with negligible body forces. This is subjected to given imposed strains $\theta_{ij}(t)$, to given mechanical loading $P_i(t)$ over part S_p of S, and to zero surface velocities over the remainder S_u of S. The imposed strains and mechanical loads both have period τ so that $\theta_{ij}(t+\tau) = \theta_{ij}(t)$ and $P_i(t+\tau) = P_i(t)$. For brevity, states of stress and strain at times t and $t+\tau$ are denoted by unstarred and starred quantities respectively, i.e.

$\sigma_{ij} \equiv \sigma_{ij}(t)$, $\sigma_{ij}^{\dagger} \equiv \sigma_{ij}(t+\tau)$, etc. Consider the positive-definite metric ρ given by

$$\rho = \frac{1}{2}\int_v \{A_{ijkl}(\sigma_{ij}^{\dagger} - \sigma_{ij})(\sigma_{kl}^{\dagger} - \sigma_{kl}) + m(p_{ij}^{\dagger} - p_{ij})(p_{ij}^{\dagger} - p_{ij})\}\,\mathrm{d}V. \tag{13}$$

The term involving plastic strains is only required for hardening materials and perfect plasticity is considered by setting the hardening coefficient $m = 0$. Differentiating eqn (13) w.r.t. time noting eqn (2)

$$\dot{\rho} = \int_v \{(\sigma_{ij}^{\dagger} - \sigma_{ij})(\dot{e}_{ij}^{\dagger} - \dot{e}_{ij}) + m(p_{ij}^{\dagger} - p_{ij})(\dot{p}_{ij}^{\dagger} - \dot{p}_{ij})\}\,\mathrm{d}V. \tag{14}$$

Since the imposed strains are cyclic, $\theta_{ij}^{\dagger} = \theta_{ij}$\{i.e. $\theta_{ij}(t+\tau) = \theta_{ij}(t)$\} and it follows from eqn (1) that

$$\dot{e}_{ij}^{\dagger} - \dot{e}_{ij} = \dot{\epsilon}_{ij}^{\dagger} - \dot{\epsilon}_{ij} - (\dot{V}_{ij}^{\dagger} - \dot{V}_{ij}) - (\dot{p}_{ij}^{\dagger} - \dot{p}_{ij}). \tag{15}$$

Since the mechanical loads are cyclic, both $\sigma_{ij}^{\dagger}$ and σ_{ij} are in equilibrium with the same loading $P_i(t)\{=P_i(t+\tau)\}$. Combination of eqns (14, 15) using the principle of virtual work then gives

$$-\dot{\rho} = \int_v \{(\sigma_{ij}^{\dagger} - \sigma_{ij})(\dot{V}_{ij}^{\dagger} - \dot{V}_{ij}) + [(\sigma_{ij}^{\dagger} - mp_{ij}^{\dagger}) - (\sigma_{ij} - mp_{ij})](\dot{p}_{ij}^{*} - \dot{p}_{ij})\}\,\mathrm{d}V. \tag{16}$$

Since there is no volume change associated with plastic deformation, for hardening materials $(\sigma_{ij} - mp_{ij})$ can be replaced by T_{ij} in eqn (16). Combining eqn (16) with the first inequality of (5) and with inequality (7) or (11) then gives $\dot{\rho} \leq 0$. However, ρ is bounded from below ($\rho \geq 0$) and hence $\dot{\rho} \to 0$ for large times. Since the creep energy dissipation surface is strictly convex, it follows from eqn (16) that, for large times, $\sigma_{ij}^{\dagger} \to \sigma_{ij}$, i.e. that $\sigma_{ij}(t+\tau) \to \sigma_{ij}(t)$. Hence a cyclic stationary state of stress is obtained. For a hardening material, it also follows that the plastic strains become cyclic, $p_{ij}(t+\tau) \to p_{ij}(t)$.

It can readily be shown that the cyclic stationary state of stress is independent of any initial state of residual stress or plastic strain in the structure. Suppose that the starred and unstarred quantities above are identified with the states at the same time $t(>0)$ in two structures which have identical histories $\theta_{ij}(t)$ and $P_i(t)$ for $t > 0$ but which have different initial states at $t = 0$. It immediately follows, as before, that $\sigma_{ij}^{\dagger} \to \sigma_{ij}$ for large times. Hence the cyclic stationary state is uniquely determined by the cycle of imposed strains and loads.

3.1 *Cyclic plasticity solutions*

In the absence of creep, the condition $\dot{\rho} \to 0$ for large times does not require $\sigma_{ij}^{\dagger} \to \sigma_{ij}$ in eqn (16) and it does not immediately follow that a cyclic stationary state is approached. However, application of virtual work and eqn (1) yields

$$\int_v (\sigma_{ij}^{\dagger} - \sigma_{ij})\{(e_{ij}^{\dagger} - e_{ij}) + (p_{ij}^{\dagger} - p_{ij})\}\,\mathrm{d}V = 0,$$

and eqn (13) simplifies to

$$\rho = \frac{1}{2}\int_v \{(\sigma_{ij}^{\dagger} - mp_{ij}^{\dagger}) - (\sigma_{ij} - mp_{ij})\}(p_{ij}^{\dagger} - p_{ij})\,\mathrm{d}V. \tag{17}$$

It further follows from a proof given in the following Section 3.2 that ultimately the stress rates become periodic ($\dot{\sigma}_{ij}^{\dagger} = \dot{\sigma}_{ij}$) and uniquely determined by the imposed cycle. In addition, for hardening materials, ultimately $\dot{T}_{ij}^{\dagger} = \dot{T}_{ij}$ and is unique.

For the perfectly-plastic materials the condition $\dot{\rho} = 0$ in eqn (16) required either $\sigma_{ij}^{\dagger} = \sigma_{ij}$ or $\dot{p}_{ij}^{\dagger} = \dot{p}_{ij} = 0$. Since the stress rates are ultimately defined, it follows that if $\dot{p}_{ij} \neq 0$ at any time t_1 $(t \leq t_1 \leq t+\tau)$ for large t, $\sigma_{ij}^{\dagger} = \sigma_{ij}$. If $\dot{p}_{ij} = 0$ for all t_1 $(t \leq t_1 \leq t+\tau)$ clearly $p_{ij}^{\dagger} = p_{ij}$. Hence either $\sigma_{ij}^{*} = \sigma_{ij}$ or $p_{ij}^{*} = p_{ij}$ and, from eqn (17), $\rho \to 0$ for large times. It then follows from eqn (13) that the stress distribution becomes periodic throughout the structure, $\sigma_{ij}(t+\tau) \to \sigma_{ij}(t)$.

For hardening materials the condition $\dot{\rho} = 0$ in eqn (16) requires either $T_{ij}^{*} = T_{ij}$ or $\dot{p}_{ij}^{*} = \dot{p}_{ij} = 0$. Arguments similar to those for perfect plasticity then require $\rho \to 0$ for large times. It then follows from eqn (13) that both the stress and plastic strain distributions become periodic throughout the structure, $\sigma_{ij}(t+\tau) \to \sigma_{ij}(t)$ and $p_{ij}(t+\tau) \to p_{ij}(t)$.

Hence, for both perfectly-plastic and hardening materials, a cyclic stationary solution is approached in the absence of creep. Such a solution, which will be termed a cyclic plasticity solution, consists of elastic zones in which no plastic strains occur and of yield zones in which periodic plastic deformation occurs. Certain properties of cyclic plasticity solutions may be deduced and these are given below for the two types of plastic behaviour.

For perfectly-plastic materials the stress distribution in the yield zones is unique. In elastic zones the stress distribution is indeterminate allowing a residual stress field constant in time. Although the plastic strain rates are periodic, the plastic strains are not necessarily periodic and may have a constant increase per cycle in cases of ratchetting. In this paper the term ratchetting is used to describe those cases for which there is a constant increase in strain per cycle in the cyclic stationary state in the absence of creep. There is no such ratchetting for hardening materials although there will be an accumulation of strain in reaching the cyclic stationary state. For perfectly-plastic materials there will be either shakedown, ratchetting or the intermediate behaviour of plastic cycling for which plastic strains occur within the cycle but there is no net increase in strain over the cycle.

For hardening materials the plastic strains are periodic throughout the structure in the cyclic stationary state so that ratchetting does not occur. The tensor T_{ij} is unique in the yield zones but the plastic strains are indeterminate allowing an arbitrary distribution of plastic strain constant in time. Consequently in both elastic and yield zones, the stress distribution is indeterminate allowing an arbitrary residual stress constant in time.

3.2 *Uniqueness of stress rates*

In this section it is shown that the stress rates for a cyclic plasticity solution are unique. A more general result, useful later, is proved: in the absence of ratchetting, for loadings that differ by only a constant mechanical load, corresponding cyclic plasticity solutions differ by only a constant stress field.

Consider two identical structures composed of identical material. One structure has imposed strains $\theta_{ij}(t)$ and applied loads $P_i(t)$ both of period τ. A cyclic plasticity solution for this structure is denoted by unstarred quantities. The second has imposed strains $\theta_{ij}(t)$ and applied loads $P_i(t) + R_i$ where R_i are constant in time. A cyclic plasticity solution for this structure is denoted by starred quantities. From the principle of virtual work

$$\begin{aligned}\int_v \int_0^\tau (\sigma_{ij}^{*} - \sigma_{ij})(\dot{\epsilon}_{ij}^{*} - \dot{\epsilon}_{ij})\,dt\,dV &= \int_s \int_0^\tau R_i(\dot{u}_i^{*} - \dot{u}_i)\,dt\,dS \\ &= \int_s R_i \int_0^\tau (\dot{u}_i^{*} - \dot{u}_i)\,dt\,dS \\ &= 0\end{aligned}$$

since ratchetting cases have been excluded. It can be seen, however, that this restriction is not needed in the case $R_i = 0$. Using eqn (1) noting that the imposed strains are the same for both structures,

$$\int_v \int_0^\tau \{(\sigma_{ij}^{*} - \sigma_{ij})(\dot{e}_{ij}^{*} - \dot{e}_{ij}) + (\sigma_{ij}^{*} - \sigma_{ij})(\dot{p}_{ij}^{*} - \dot{p}_{ij})\}\,dt\,dV = 0$$

$$\therefore \int_v \int_0^\tau (\sigma_{ij}^{*} - \sigma_{ij})(\dot{p}_{ij}^{*} - \dot{p}_{ij})\,dt\,dV = 0 \tag{18}$$

since both stress fields are cyclic. In the case of a hardening material, the plastic strains are also cyclic so that eqn (18) can be replaced by

$$\int_v \int_0^\tau (T_{ij}^* - T_{ij})(\dot{p}_{ij}^* - \dot{p}_{ij})\, dt\, dV = 0. \tag{18a}$$

For a perfectly-plastic material, the restrictions on equality in relationship (7) when combined with eqn (18) lead to the result that either $\sigma_{ij}^* = \sigma_{ij}$ or $\dot{p}_{ij}^* = \dot{p}_{ij} = 0$. For hardening materials the restrictions on equality in relationship (11) when combined with eqn (18a) require that either $T_{ij}^* = T_{ij}$ or $\dot{p}_{ij}^* = \dot{p}_{ij} = 0$.

Both structures are subjected to the same rate of mechanical loading $\dot{P}_i(t)$ so that the principle of virtual work gives

$$\int_v (\dot{\sigma}_{ij}^* - \dot{\sigma}_{ij})(\dot{\epsilon}_{ij}^* - \dot{\epsilon}_{ij})\, dV = 0$$

i.e.

$$\int_v \{(\dot{\sigma}_{ij}^* - \dot{\sigma}_{ij})(\dot{e}_{ij}^* - \dot{e}_{ij}) + (\dot{\sigma}_{ij}^* - \dot{\sigma}_{ij})(\dot{p}_{ij}^* - \dot{p}_{ij})\}\, dV = 0. \tag{19}$$

For a hardening material eqn (19) may be written

$$\int_v \{(\dot{\sigma}_{ij}^* - \dot{\sigma}_{ij})(\dot{e}_{ij}^* - \dot{e}_{ij}) + m(\dot{p}_{ij}^* - \dot{p}_{ij})(\dot{p}_{ij}^* - \dot{p}_{ij}) + (\dot{T}_{ij}^* - \dot{T}_{ij})(\dot{p}_{ij}^* - \dot{p}_{ij})\}\, dV = 0. \tag{19a}$$

However, it has been shown that $\dot{p}_{ij}^* = \dot{p}_{ij} = 0$ unless $\sigma_{ij}^* = \sigma_{ij}$ for perfect-plasticity or unless $T_{ij}^* = T_{ij}$ for hardening. Inequalities (8, 12) can then be applied so that eqns (19) and (19a) reduce to

$$\int_v (\dot{\sigma}_{ij}^* - \dot{\sigma}_{ij})(\dot{e}_{ij}^* - \dot{e}_{ij})\, dV \le 0.$$

The only possible solution of this is that $\dot{\sigma}_{ij}^* = \dot{\sigma}_{ij}$ which is the required result. A further consequence of (19a) is that for hardening materials the plastic strain rates are also unique ($\dot{p}_{ij}^* = \dot{p}_{ij}$).

Equations (19) and (19a) are still valid if the starred and unstarred quantities are identified, as earlier, with states at times $(t + \tau)$ and t in the same structure. The condition $\dot{\rho} = 0$ in eqn (16) requires that $\dot{p}_{ij}^* = \dot{p}_{ij}$ unless $\sigma_{ij}^* = \sigma_{ij}$ for perfect plasticity or unless $T_{ij}^* = T_{ij}$ for hardening. Hence, as above, $\dot{\sigma}_{ij}^* = \dot{\sigma}_{ij}$, i.e. $\dot{\sigma}_{ij}(t + \tau) = \dot{\sigma}_{ij}(t)$ for large times, t. The conditions that the stress rates are periodic and unique were those used in Section 3.1 to demonstrate that a cyclic stationary state is approached.

4. UPPER BOUNDS FOR THE CYCLIC SOLUTION

In this section cyclic plasticity solutions are used to provide upper bound estimates of the displacements, work and creep energy dissipation associated with the cyclic stationary state of a creeping structure. Consider two identical structures composed of materials with identical elastic-plastic behaviour. One structure has an applied loading $P_i(t)$ and imposed strains $\theta_{ij}(t)$ both of period τ. The material of this structure creeps and the cyclic stationary state of stress is denoted by $\sigma_{ij}(t)$ with corresponding strains and displacements denoted by unstarred quantities. The second structure has an applied loading $P_i(t) + R_i(t)$ where $R_i(t)$ is periodic of period τ, and imposed strains $\theta_{ij}(t)$. The material of this structure does not creep. A cyclic plasticity solution is denoted by $\sigma_{ij}^*(t)$ with corresponding strains and displacements denoted by starred quantities. From the principle of virtual work,

$$\int_0^\tau \int_s R_i(\dot{u}_i - \dot{u}_i^*)\, ds\, dt = \int_0^\tau \int_v (\sigma_{ij}^* - \sigma_{ij})(\dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^*)\, dV\, dt.$$

Splitting the total strains into components using eqn (1),

$$\int_0^\tau \int_s R_i(\dot{u}_i - \dot{u}_i^\dagger)\,ds\,dt = \int_0^\tau \int_v (\sigma_{ij}^\dagger - \sigma_{ij})\{\dot{e}_{ij} - \dot{e}_{ij}^\dagger + \dot{p}_{ij} - \dot{p}_{ij}^\dagger + \dot{V}_{ij}\}\,dV\,dt$$

$$= \int_0^\tau \int_v (\sigma_{ij}^\dagger - \sigma_{ij})(\dot{p}_{ij} - \dot{p}_{ij}^\dagger + \dot{V}_{ij})\,dV\,dt \tag{20}$$

since both stress fields are periodic. For a hardening material, it has been shown in Section 3 that the plastic strains are cyclic so that (20) may be written

$$\int_0^\tau \int_s R_i(\dot{u}_i - \dot{u}_i^\dagger)\,ds\,dt = \int_0^\tau \int_v \{(T_{ij}^\dagger - T_{ij})(\dot{p}_{ij} - \dot{p}_{ij}^\dagger) + (\sigma_{ij}^\dagger - \sigma_{ij})\dot{V}_{ij}\}\,dV\,dt. \tag{20a}$$

Combination of eqns (7), (20) or (11) and (20a) with the third inequality of (5) gives

$$\int_0^\tau \int_s R_i \dot{u}_i\,ds\,dt \le \int_0^\tau \int_s R_i \dot{u}_i^\dagger\,ds\,dt + \frac{1}{n}\left(\frac{n}{n+1}\right)^{n+1} \int_0^\tau \int_v \dot{D}(\sigma_{ij}^\dagger/\sigma_0)\,dV\,dt. \tag{21}$$

For imposed load and strain variations which are within the shakedown limit, $\sigma_{ij}^\dagger$ is an elastic cyclic solution and inequality (21) reduces to the result of Ponter[2].

4.1 *Point displacement bound*

If the additional loads R_i are taken as a point load $R (R > 0)$ which is constant in time, inequality (21) bounds the displacement U_R in the line of R, as

$$U_R(\tau) - U_R(0) \le U_R^\dagger(\tau) - U_R^\dagger(0) + \frac{1}{nR}\left(\frac{n}{n+1}\right)^{n+1} \int_0^\tau \int_v \dot{D}(\sigma_{ij}^\dagger/\sigma_0)\,dV\,dt. \tag{22}$$

The term $\{U_R^\dagger(\tau) - U_R^\dagger(0)\}$ is only non-zero for perfectly-plastic materials when the loading is sufficient to cause ratchetting.

4.2 *Work bound*

Setting the additional loads proportional to the applied loads as $R_i = (\mu - 1)P_i$ where μ is a constant ($\mu > 1$), gives a work bound from (21) as

$$\int_0^\tau \int_s P_i \dot{u}_i\,ds\,dt \le \int_0^\tau \int_s P_i \dot{u}_i^\dagger\,ds\,dt + \frac{1}{(\mu - 1)n}\left(\frac{n}{n+1}\right)^{n+1} \int_0^\tau \int_v \dot{D}(\sigma_{ij}^\dagger/\sigma_0)\,dV\,dt. \tag{23}$$

The first term on the right hand side of inequality (23) represents inelastic work.

4.3 *Creep energy dissipation bound*

If the additional loads R_i are set equal to zero, then combination of eqns (7, 20) or (11, 20a) with the second inequality of (5) gives

$$\int_0^\tau \int_v \dot{D}(\sigma_{ij}/\sigma_0)\,dV\,dt \le \int_0^\tau \int_v \dot{D}(\sigma_{ij}^\dagger/\sigma_0)\,dV\,dt \tag{24}$$

which is a bound on the mean creep energy dissipation rate.

5. OPTIMISATION OF THE BOUNDS

It was noted in Section 3.1 that the stress distribution for a cyclic plasticity solution is not unique. The bounds (22)–(24) can then be optimised by suitable choice of any additional loads (R or μ) and by choice of the stress distribution $\sigma_{ij}^\dagger$. For perfectly-plastic materials, choice of $\sigma_{ij}^\dagger$ is constrained by the yield criterion but there is no such restriction for hardening materials because the yield criterion can always be satisfied by suitable choice of the plastic strain field.

Consider, as an example of optimisation, the point displacement bound (22) in the absence of ratchetting,

$$U_R(\tau) - U_R(0) \le \frac{1}{nR}\left(\frac{n}{n+1}\right)^{n+1} \int_0^\tau \int_v \dot{D}(\sigma_{ij}^\dagger/\sigma_0)\,\mathrm{d}V\,\mathrm{d}t. \tag{25}$$

Introducing Lagrangian multipliers $\dot{\Lambda}$ for which

$$\dot{\Lambda} \ge 0, \quad f(\sigma_{ij}^\dagger) = 0; \quad \dot{\Lambda} = 0, \quad f(\sigma_{ij}^\dagger) < 0 \tag{26}$$

consider the problem of minimising

$$W = \frac{1}{R}\int_0^\tau \int_v \{\dot{D}(\sigma_{ij}^\dagger/\sigma_0) + (n+1)\dot{\Lambda} f(\sigma_{ij}^\dagger)\}\,\mathrm{d}V\,\mathrm{d}t. \tag{27}$$

From the theorems of Kuhn and Tucker[13], at the minimum $\delta W = 0$, hence using the definition of $\dot{D}$ given by eqn (3),

$$W\delta R = (n+1)\int_0^\tau \int_v [\dot{V}_0 \phi^n\{\partial\phi/\partial(\sigma_{ij}^\dagger/\sigma_0)\}g(\theta) + \dot{\Lambda}\partial f/\partial\sigma_{ij}^\dagger]\delta\sigma_{ij}^\dagger\,\mathrm{d}V\,\mathrm{d}t.$$

Since the load variation δR is constant in time, the results of section 3.2 require $\delta\sigma_{ij}^\dagger$ to be independent of time. Consequently

$$W\delta R = (n+1)\int_v \int_0^\tau [\dot{V}_0 \phi^n\{\partial\phi/\partial(\sigma_{ij}^\dagger/\sigma_0)\}g(\theta) + \dot{\Lambda}\partial f/\partial\sigma_{ij}^\dagger]\,\mathrm{d}t\,\delta\sigma_{ij}^\dagger\,\mathrm{d}V. \tag{28}$$

Since $\delta\sigma_{ij}^\dagger$ must be in equilibrium with the load variation δR, the principle of complementary virtual work requires

$$\int_0^\tau [\dot{V}_0 \phi^n\{\partial\phi/\partial(\sigma_{ij}^\dagger/\sigma_0)\}g(\theta) + \dot{\Lambda}\partial f/\partial\sigma_{ij}^\dagger]\,\mathrm{d}t \tag{29}$$

to be kinematically admissible with a displacement in the line of R equal to $W/(n+1)$. Comparison of (29) with eqns (4) and (6) and the restraints (26) on $\dot{\Lambda}$, enables the following physical interpretation: The optimum cyclic plasticity solution is one for which the corresponding creep strain rates plus possible plastic strain rates are kinematically admissible when integrated over the cycle. It should be noted that the possible plastic strain rates of (29) are not directly related to the actual plastic strain rates $\dot{p}_{ij}^\dagger$. For any cyclic plasticity solution $\sigma_{ij}^\dagger$ the quantity (29) will be termed the "associated strain cycle".

Denoting as Δu^* the displacement in the line of R derived for the strain field (29), eqn (28) requires

$$W = (n+1)\Delta u^* \tag{30}$$

and the original displacement bound (25) may be written

$$U_R(\tau) - U_R(0) \le \left(\frac{n}{n+1}\right)^n \Delta u^*.$$

For loading below the shakedown limit, these optimum conditions have been given by Ponter[14].

For hardening materials, the yield criterion does not constrain the minimisation so that the associated strain cycle (29) consists entirely of creep strains corresponding to $\sigma_{ij}^\dagger$. It has been shown in Section 3.2 that the plastic strain rates are unique for a hardening material and it readily follows that all bounds (22)–(24) are optimised by using the cyclic plasticity solution with a kinematically admissible associated strain cycle.

For perfectly-plastic materials, the associated strain cycle is kinematically admissible for the optimum creep energy dissipation bound (24). The displacement and work bounds (22) and (23) are optimised by making the associated strain cycle kinematically admissible if the ratchet displacements and inelastic work are a function only of the loading. Cases (such as plastic collapse) where displacements are indeterminate are not considered. If the displacements cannot be evaluated in the absence of creep, the additional displacements due to creep are of little practical importance.

The condition that the associated strain cycle should be kinematically admissible can be used to determine the optimum cyclic plasticity solution by the methods described in [4]. Conditions such as (30) can be used to obtain rapid convergence to the optimum bound and this is illustrated in the analysis of simple structures given in the companion paper[5].

5.1 *Uniqueness of optimum solution*

It is now shown that, for given loading conditions, the stress distribution which has a kinematically admissible associated strain cycle is unique. Suppose that the converse is true and there exist two cyclic plasticity solutions σ_{ij}, $\sigma_{ij}^*(\sigma_{ij} \neq \sigma_{ij}^*)$ both of which have kinematically admissible associated strain cycles. The conditions (26) and the convexity of the yield surface require, in similar manner to inequality (7),

$$(\sigma_{ij}^* - \sigma_{ij})(\dot{\lambda}\,\partial f/\partial\sigma_{ij}^* - \dot{\lambda}\,\partial f/\partial\sigma_{ij}) \geq 0. \tag{31}$$

From virtual work,

$$\int_v (\sigma_{ij}^* - \sigma_{ij}) \int_0^\tau [\dot{V}_0 g(\theta)\{\phi^n(\sigma_{ij}^*)\partial\phi/\partial\sigma_{ij}^* - \phi^n(\sigma_{ij})\partial\phi/\partial\sigma_{ij}\} + \dot{\lambda}^*\partial f/\partial\sigma_{ij}^* - \dot{\lambda}\partial f/\partial\sigma_{ij}]\,\mathrm{d}t\,\mathrm{d}V = 0,$$

since both associated strain cycles are kinematically admissible. From the results of Section 3.2 the difference $(\sigma_{ij}^* - \sigma_{ij})$ is independent of time and hence can be brought within the time integral. Use of inequality (31) then gives

$$\int_v \int_0^\tau (\sigma_{ij}^* - \sigma_{ij})\{\phi^n(\sigma_{ij}^*)\partial\phi/\partial\sigma_{ij}^* - \phi^n(\sigma_{ij})\partial\phi/\partial\sigma_{ij}\}g(\theta)\,\mathrm{d}t\,\mathrm{d}V \leq 0.$$

However, from the strict convexity of the creep dissipation surface, the integrand of this inequality is positive unless $\sigma_{ij}^* = \sigma_{ij}$. Hence $\sigma_{ij}^* = \sigma_{ij}$ everywhere at all times, the original assumption $\sigma_{ij}^* \neq \sigma_{ij}$ is violated and the cyclic plasticity solution with a kinematically admissible associated strain cycle is unique.

5.2 *Approximate solutions*

In Ainsworth[4], for loading within the shakedown limit, the cyclic elastic solution with a kinematically admissible associated strain cycle was considered an approximate solution. The approximation is better for shorter cycle times and may be considered exact in the limit $\tau \to 0$. Similarly, for loading beyond the shakedown limit, the cyclic plasticity solution which has a kinematically admissible associated strain cycle may be considered an approximate or limiting solution. In view of the bound (24), such a solution provides a safe estimate of the creep energy dissiption occurring for any actual solution. As such, although it cannot guarantee to give a safe estimate of any particular displacement, the approximate solution provides a safe estimate of the mean strains occurring in a structure. Since there are no additional loads for which to optimise, it is slightly easier to evaluate than the upper bounds on work and displacement.

6. CONCLUDING REMARKS

For load variations above the shakedown limit, the properties of cyclic plasticity solutions have been examined allowing yield criteria of perfect-plasticity and kinematic strain hardening. The cyclic plasticity solutions have been used to provide upper bounds on the work, displacement and creep energy dissipations which occur in the cyclic stationary state of a creeping structure. This extends previous work of Ponter[1] to the area above the shakedown limit and hence enables simplified methods to be applied to a wider class of problem.

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